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Velocity Operators and Time–Energy Relations in Relativistic Quantum Mechanics

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According to both Dirac's and Kemmer's relativistic quantum theories, the eigenvalues of the velocity operator are $+c$ and $-c$. This false result is avoided if certain alternative particle coordinates are adopted. Another advantage is that the new coordinates occur in additional constants of the motion. These are sui generis angular momenta obtained by taking the vector product of the nonstandard coordinates with the linear momentum. An additional virtue of the new velocity operator is that, like in classical mechanics, it is proportional to the linear momentum. Besides, the zeroth component of the new set of coordinates does not commute with the hamiltonian, which results in a genuine "indeterminacy" relation between time and energy.

KEY WORDS: momentum; velocity; time–energy relation; relativistic quantum mechanics.

1. INTRODUCTION

The standard velocity operator in Dirac's theory of electrons is $dx/dt = -c\alpha$. This operator is inadequate for two reasons: First, because its eigenvalues are $+c$ and $-c$, where c is the speed of light in a vacuum; and second, because it is not proportional to the linear momentum. Similar results hold in Kemmer's theory for particles with spin 0 or 1. We submit that these results follow from the assumption that the spatial coordinate x is made to double as the particle position coordinate.

An alternative in the case of Dirac's theory is to adopt $X^{\mu} = x^{\mu} + (\Lambda/2)\, i\gamma^{\mu}$ as the position operator, where $\Lambda = (h/2\pi m_0 c)$ is the Compton wavelength. The vector product of this operator with the linear momentum yields an interesting, if neglected, set of constants of the motion. Our hypothesis entails that the electron velocity is proportional to the momentum, just as in classical particle mechanics. O'Connell and Wigner (1977) too retrieved the classical relation between velocity and momentum, though using the somewhat unwieldly coordinate position introduced earlier by Newton and Wigner (1949).

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The counterparts in Kemmer's theory can likewise be repaired by redefining the position coordinate and the associated velocity operator in a similar fashion. It turns out that in this case, too, a semiclassical relation between velocity and momentum results.

Besides, the zeroth components of the new position coordinates lead to genuine zeroth Heisenberg inequalities involving the mean standard deviations of the energy and the said time coordinates.

2. INADEQUACY OF THE STANDARD VELOCITY OPERATOR FOR THE ELECTRON

As is well-known, in Dirac's theory of the electron the relativistic velocity operator is $dx/dt = -c\alpha$, where the hypercomplex "number" α is the coefficient of the momentum in Dirac's hamiltonian $H_{\mathrm{D}} = c\alpha p + m_0c^2\beta$. The three α's and β are representable by 4×4 matrices. The eigenvalues of each of the three components of $-c\alpha$ are $+c$ and $-c$. Besides, these components do not commute with one another. Consequently, they have no precise values at the same time—whence dx/dt cannot be said to be a vector, let alone a measurable one.

This result violates the relativistic ban on luminal velocities for entities endowed with mass. It is also at variance with the principle (or dogma) that the eigenvalues of a dynamical variable are the possible outcomes of measurements of it. These difficulties are sometimes justified by saying that any measured value of the velocity is the average of a variable that oscillates between $+c$ and $-c$. However, this is a lame excuse. Why not admit that the formula in question is not just "very remarkable," as Bethe (1964, p. 205) commented, but physically meaningless?

If "$dx/dt = -c\alpha$" is indeed meaningless, then we must identify the false premise(s) that entail(s) it. The obvious suspect is the tacit assumption that x is both the label identifying a point in space and the particle coordinate. Surely this assumption is false, for the values of x depend only on the coordinate system, whereas those of a genuine particle coordinate depend also upon the particle's state. In other words, x is not a dynamical variable on a par with the linear momentum and the spin: it is a geometric coordinate, not a physical one. Decorating it with a cap will not do, because in the Schrödinger representation, contrary to the Heisenberg one, position is not a hermitian operator but an ordinary real variable.

One way of saving the formula under discussion might be to suppose that it represents the spatial disturbance caused by the presence of the electron. This disturbance could be regarded as the superposition of two shock waves, or spatial ripples, one with speed $+c$, and the other with speed $-c$, in the x-direction. This assumption is consistent with the hypothesis that the line element in the putative space attached to the electron is $ds = \gamma_\mu dx^\mu$, where $\gamma_\mu = \beta\alpha_\mu$, and $\alpha_0 = I$. Indeed, $ds^2 = dx^\mu dx_\mu$. Thus, unlike the underlying pseudo-Euclidean space, which is rigid, the electron space would be fluctuating or jelly-like.

In other words, the *Zitterbewegung* (trembling motion), first described by Schrödinger in 1930, would be a property of the space surrounding the electron, rather than a kinematical feature of the electron itself. But this speculation, however intriguing, will remain a fantasy as long as the geometry of the putative electron space and its possible experimentally testable features are not further investigated. And neither task could reasonably be accomplished independently of quantum electrodynamics, if only because this theory involves the fluctuating electrodynamic vacuum. Let us therefore return to our initial question: Which is the correct velocity operator?

Several answers to this question have been proposed. One of them is that of Foldy and Wouthuysen (1950). These authors altered the standard representation of the Dirac matrices, and subjected x to a unitary transformation. They thus obtained a plausible result with a meaningful classical limit (Costello and McKellar, 1995). However, unlike ours, their position coordinate is not Lorentz-covariant. Besides, far from having an obvious physical meaning, the Foldy–Wouthuysen transformation looks like an ad hoc afterthought. We prefer to question the tacit assumption that x is both a geometric and a physical coordinate. We will adopt a nonstandard particle coordinate that is physically meaningful because it is anchored to a constant of the motion. And we will do something similar for Kemmer's theory.

3. ALTERNATIVE POSITION AND VELOCITY OPERATORS FOR SPIN 1/2 PARTICLES

The question of the correct velocity operator presupposes the answer to another question, namely: Which is the adequate particle coordinate? In the following we propose to explore the following tentative answer: The correct position coordinate for a fermion, such as an electron, is the operator

$$X^\mu = x^\mu + (\Lambda/2)\, i\gamma^\mu, \quad \text{where} \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\delta\mu\nu \quad \text{and} \quad \Lambda = h/(2\pi m_0 c). \tag{1a}$$

The eigenvalues of this operator are

$$\text{eiv}X^\mu = x^\mu \pm (\Lambda/2).$$

Recalling that $dx^\mu/ds = \gamma^\mu$, where s designates the proper time, (1a) can be rewritten as

$$X^\mu = x^\mu + (\Lambda/2)i(dx^\mu/ds). \tag{1b}$$

This shows that X^μ is manifestly Lorentz covariant. Obviously, the difference between the two coordinates disappears in the nonquantum limit $h = 0$.

Interestingly, X^μ is not an ad hoc contrivance but a physically meaningful operator, because it occurs in the nonstandard angular momentum

$$M_\nu^\mu = X^\mu \Pi_\nu - X^\nu \Pi_\mu, \quad \text{where} \quad \Pi_\mu = p_\mu - (e/c)A_\mu, \tag{2}$$

A_μ is the vector potential that generates the electromagnetic field intensities. Each of the six components of this tensor is a constant of the motion of an electron in a central field and, a fortiori, of a free electron (Bunge, 1955; Bunge and Kálnay, 1969). We submit that, since conservation equations are physical laws par excellence, X^μ is a bona fide dynamical variable.

A quick calculation yields the velocity associated with the spatial component X of X^μ:

$$V = dX/dt = -(1/m_0)\beta p. \tag{3}$$

The annoying term $-c\alpha$ originating in x has been compensated for by a term $+c\alpha$ originating in γ. The new velocity operator is thus proportional to the linear momentum—as it should be in accordance with the Einstein–Bohr correspondence principle about rival theories. The two vectors in question are parallel in the case of the electron (represented by the two -1 entries of β), and antiparallel in that of the positron (represented by the $+1$ entries of β). And since the components of dX/dt commute with one another, it is a vector, unlike dx/dt.

The nonquantum analog of (3) is the relativistic formula

$$v = (1 - v^2/c^2)^{1/2}(p/m_0).$$

However, the classical analogy stops right here. Indeed, the particle is accelerated even when free. In fact,

$$d^2Xdt^2 = -(4\pi c/hm_0)(i\gamma p)p. \tag{4}$$

Thus, X too suffers from *Zitterbewegung*, though less severely so than x, since dX/dt is diagonal and therefore does not fluctuate. Moreover, d^2X/dt^2 shares with dx/dt and d^2x/dt^2 the undesirable property that its components fail to commute with one another. Hence, unlike dX/dt, the acceleration is not a vector.

There are further disanalogies with both relativistic mechanics and nonrelativistic quantum mechanics. To begin with, the components of X do not commute among themselves. For example,

$$[X_1, X_2] = -i(\Lambda^2/2)\sigma_3. \tag{5}$$

According to the standard (or Copenhagen) interpretation, this equation, like any of its two partners, means that a precise measurement of any of the position components precludes an exact measurement of the other two. And according to the realist interpretation, (5) means that the three components of X do not exist at the same time: The electron is not precisely localized. Consequently its position cannot be measured exactly, and this regardless of the Heisenberg inequality. On either interpretation, X does not qualify as a vector. This result shows that Dirac's theory does not include a kinematics. More on this below.

Secondly, X and V do not commute with one another either. Indeed,

$$[X_i, V_j] = -i(\Lambda c)\delta_{ij} + i(\Lambda/m_0)\alpha_i p_j, \qquad i, j = 1, 2, 3. \tag{6}$$

Recall now that, if A and B are dynamical variables with a commutator $[A, B] = iC$, then their mean standard deviations ΔA and ΔB are related by $\Delta A \cdot \Delta B \geq 2^{-1}|\langle C\rangle|$ where $\langle C\rangle$ is the expectation value of C for the given state. The Eq. (6) for $i = j$ leads to the following relativistic generalization of Heisenberg's inequalities for spin one-half particles:

$$\Delta X_i \cdot \Delta V_i \geq (h/4\pi m_0) + (\Lambda/2m_0)| < \alpha_i p_i > |, \qquad i = 1, 2, 3, \tag{7a}$$

where $| < \alpha_i p_i > |$ is the absolute value of the contribution of the ith component to the total average kinetic energy.

The first term of the RHS of (7a) is similar to the nonrelativistic result. By contrast, the second term in the RHS of (7a) is an energy-dependent correction, as might have been expected.

The Eq. (6) for $i \neq j$ leads to

$$\Delta X_i \cdot \Delta V_j \geq (\Lambda/2m_0)| < \alpha_i p_j > |, \qquad i, j = 1, 2, 3, \quad i \neq j. \tag{7b}$$

These inequalities lack nonrelativistic limits. They suggest that, on the relativistic theory, the corpuscular aspect is even more blurred than on the nonrelativistic theory.

4. THE CASE OF KEMMER'S THEORY FOR SPINS 0 AND 1

Kemmer's theory was initially proposed to describe spin 0 and 1 particles. Though rarely used nowadays, it is still interesting if only because its mathematical formalism is richer than Dirac's. The basic Kemmer equation is typographically almost identical with Dirac's:

$$\left[\beta_\mu \partial x^\mu + (1/\Lambda)\right]\Psi = 0, \qquad \text{with} \quad \mu = 1, 2, 3, 4. \tag{5}$$

But the coefficients of the four-momentum are now implicitly defined by

$$\beta_\lambda \beta_\mu \beta_\nu + \beta_\nu \beta_\mu \beta_\lambda = \delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\mu}\beta_\nu. \tag{6}$$

These hypercomplex "numbers" can be represented by 16×16 singular matrices, whose eigenvalues are 0, 1, and -1.

To calculate time derivatives, we need the Kemmer hamiltonian. This can be written as

$$H_{\mathrm{K}} = c\alpha p + m_0 c^2 \beta_0, \tag{7}$$

where

$$\alpha_K = i(\beta_4\beta_K - \beta_K\beta_4), \qquad \beta_4 = i\beta_0. \tag{8}$$

We now assume that the particle position is represented by

$$X^\mu = x^\mu - \Lambda\beta^\mu. \tag{9}$$

The eigenvalues of this operator are 0, $+\Lambda$, and $-\Lambda$. Interestingly, the vector product of this position coordinate and the linear momentum yields a new angular momentum that happens to be a constant of the motion for a free particle (Bunge, 1958):

$$M^\mu_\nu = X^\mu p_\nu - X^\nu p_\nu, \qquad dM^\mu_\nu/dt = 0. \tag{10}$$

The time derivative of the spatial coordinate has again an undesirable value, namely

$$dx/dt = -c\alpha_k, \tag{11}$$

where the Kemmer alphas are given by (8). Again, this cannot be the suitable velocity operator for a boson endowed with mass, since its eigenvalues are 0, $+c$, and $-c$, regardless of the particle state. On the other hand, the time derivative of the spatial part of (9) is similar to the Dirac case, namely

$$dX/dt = -(1/m_0)\beta_0 p, \tag{12}$$

whose eigenvalues are 0, p/m_0, and $-p/m_0$. Besides, the components of dX/dt commute with one another, so that this is a genuine vector, just as for fermions. These are certainly desirable features of X.

By contrast, the acceleration of a free Kemmer particle is

$$d^2X/dt^2 = (2\pi c/hm_0)(\beta p)p, \tag{13}$$

which is a fluctuating variable.

5. TIME AND ENERGY

With the help of the clock-in-a-box thought experiment that Bohr (1949) employed in his famous discussion with Einstein, he claimed that time and energy are canonically conjugate variables, and thus mutually complementary by analogy to position and linear momentum. However, his formula "$\Delta t \cdot \Delta E \geq h/4\pi$" is bogus, because the time variable occurring in it is what Dirac called a c number; that is, its scatter is nil (Bunge, 1970).

Let see what happens with the zeroth components of our alternative coordinates, namely

$$X^0 = x^0 + (\Lambda/2)\, i\gamma^0 \qquad \text{in Dirac's theory,} \tag{13}$$

and

$$X^0 = x^0 - \Lambda\beta^0 \qquad \text{in Kemmer's theory.} \tag{14}$$

Here $x^0 = ct$ is the "public" time, a "c-number," whereas the second term represents the internal one that "ticks" only via the state function. It turns out that

$$[X^0, H_{\mathrm{D}}] = i\Lambda c(\gamma p) \qquad \text{in Dirac's theory,} \tag{15a}$$

and

$$[X^0, H_{\mathrm{K}}] = i2\Lambda c(\beta p) \qquad \text{in Kemmer's theory.} \tag{15b}$$

Consequently, the corresponding zeroth Heisenberg inequalities are

$$\Delta X^0 \cdot \Delta E \geq (\Lambda c/2)| < \gamma p > | \qquad \text{in Dirac's theory,} \tag{16a}$$

and

$$\Delta X^0 \cdot \Delta E \geq \Lambda c| < \beta p > | \qquad \text{in Kemmer's theory.} \tag{16b}$$

These inequalities have no counterparts in nonrelativistic quantum mechanics. However, in the simplest case, when the "small" components of the corresponding spinors are negligible, the indicated averages values vanish, and we are left with

$$\Delta X^0 \cdot \Delta E \geq 0. \tag{17}$$

That is, the lower bound of the product of the "indeterminacies" or "uncertainties" (average scatters) is zero. In other words, in this case time and energy may not be mutually "complementary." However, the product in question is greater than zero if the small components of the state spinor are significant, that is, for high energies.

6. CONCLUDING REMARKS

Quantum mechanics is counterintuitive enough: There is no need to add it blatantly meaningless formulas. One of these, "$dx/dt = -c\alpha$," occurs in both Dirac's and Kemmer's theories—though with somewhat different meanings. This is a consequence of the doubtful assumption that x, in addition to labeling an arbitrary point in space, represents the position coordinate of the particle in question.

We have introduced alternative operators that involve the physical constant Λ and that do not have that undesirable feature. Besides, they occur in two new sets of constants of the motion, so that they are not ad hoc tricks devised to remove implausible results. In particular, the corresponding velocity turns out to be proportional to the momentum. Moreover, in both theories the components of the velocity commute with one another, so that they are genuine vectors.

However, the *Zitterbewegung*, which gets ironed out in the velocity, reappears in the acceleration. This shows that relativistic quantum mechanics, whether for fermions or for bosons, does not contain a kinematics, any more than its nonrelativistic counterpart does. On the other hand, genuine zeroth Heisenberg inequalities hold for the energy and time operators for particles of both kinds.

In sum, we have introduced nonstandard particle coordinates and their rates of change for both the Dirac and the Kemmer particles, that are less flawed than the standard formula "$dx/dt = -c\alpha$." Furthermore, those cooordinates are involved in interesting constants of the motion, and their zeroth components are involved in genuine "indeterminacy" inequalities. Still, our coordinates have a defect of their own: The accelerations are not nil even in the absence of external forces. That is, both theories violate the principle of inertia.

An upshot of the preceding is that the theories in question do not help compute trajectories. Hence we have not yet hit on the correct velocity operator. But it is also possible that the question of the correct velocity operator is pointless, because the concepts of velocity and acceleration might be just as alien to the quantum theory as that of trajectory. After all, the concepts of particle and wave "have only the validity of analogies which are accurate only in limiting cases" (Heisenberg, 1930, p. 10). The practical business of quantum "mechanics" is to calculate energy spectra, scattering cross-sections, and the like, not orbits. Hence, some misunderstandings of it might be avoided by renaming it "quantics" (Lévy-Leblond and Balibar, 1984) and by calling its referents "quantons" (Bunge, 1967).

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